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Note on shape oscillations of bubbles

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By use of a virial equation introduced in a recent paper (Benjamin 1987), the main results of a second-order perturbation theory developed by Longuet-Higgins (1989*a*) are recovered in comparatively simple fashion. Asymmetric capillary vibrations of a gas bubble in an infinite incompressible liquid are confirmed to generate an increase in the volume of the bubble, a lowering of the mean pressure of the gas and a monopole component in the motion of the liquid. It is shown that the second effect remains when the bubble is incompressible.

1. Introduction

In two contemporary papers Longuet-Higgins (1989*a*, *b*) investigates second-order effects that accompany shape oscillations of a gas-filled bubble in an infinite liquid. On the basis of ideal-fluid theory he shows how free oscillations at frequency σ_n in a normal mode $(n = 2, 3, \ldots)$ cause fluctuations of the bubble volume at frequency $2\sigma_n$, thus generating a monopole component in the motion of the surrounding liquid. Because the latter component decays with radial distance r as r^{-1}, in contrast with $r^{-(n+1)}$ for the asymmetric component, this second-order effect is suggested by Longuet-Higgins to be a possibly significant source of underwater sound.

In the present note the central results in his first paper (Longuet-Higgins 1989*a*; henceforth referred to as LH) will be recovered by an analytical method that is simpler in detail than his. The new method has other advantages, and a closely related application noted previously (Benjamin 1987, p. 358) will be expanded. No commentary will be made here about the possible bearing on underwater sound, which prospect is examined in the second of Longuet-Higgins's papers (1989*b*).

The present derivation depends on an *exact* virial equation exhibited in a recent paper about Hamiltonian theory for motions of a bubble in an infinite liquid (Benjamin 1987; henceforth referred to as B). Let Φ denote the evaluation of the velocity potential ϕ at the surface of the bubble (B, p. 351); let $\boldsymbol{X}$ denote the position vector from a fixed (interior) point to the surface; and let $\boldsymbol{n}$ denote the unit normal directed from the surface into the surrounding liquid. Also take the liquid to have unit density. Then the scalar describable as the virial of the motion is defined by

$$W = -\int_S \Phi(\boldsymbol{X}\cdot\boldsymbol{n})\,\mathrm{d}s \tag{1}$$

(B, equation (2.19)). The equation in question may be written

$$\frac{\mathrm{d}W}{\mathrm{d}t} = 5K - 2T|S| + 3(P_\mathrm{i} - p_\infty)\mathscr{V}, \tag{2}$$

where K is the kinetic energy of the motion, T the coefficient of surface tension (σ in B) and $|S|$ the area of the bubble surface, P_i the pressure of the gas contents and

p_∞ the pressure at infinity in the liquid ($P_i - p_\infty = P$ in B), and $\mathscr{V}$ the volume of the bubble. The proof of (2) is straightforward on lines indicated in B (pp. 357, 358). In it the inertia of the gas contents is ignored, the bubble is supposed to be simply connected, and the motion of the incompressible inviscid liquid is supposed to be generated from rest by conservative forces, so that ϕ is for all time t a harmonic function of position everywhere in the unbounded domain exterior to S. There is no other assumption.

A second general property will also be helpful, being even easier to prove from the fact that ϕ is a harmonic function. Let $A_0(t)$ denote the monopole coefficient of the far field (i.e. $\phi \sim A_0 r^{-1} + O(r^{-2})$ as $r \to \infty$). We then have the kinematic identity

$$\frac{\mathrm{d}\mathscr{V}}{\mathrm{d}t} = \int_\infty \frac{\partial \phi}{\partial r} \mathrm{d}s = -4\pi A_0 \tag{3}$$

(B, p. 356), whose physical interpretation regarding mass conservation is obvious. This identity highlights that a monopole component of the motion can only arise from changes in the bubble's volume. It cannot arise in, for example, the otherwise closely comparable case of shape oscillations executed by a drop of another liquid immersed in and immiscible with the infinite liquid.

The exact properties (2) and (3) were introduced in B as items in a list of conservation laws linked to symmetries of the Hamiltonian system composed by the hydrodynamic problem. Several useful applications of the various conservation laws and their variational ramifications were explained in B (§§4, 5), and others will be reported in due course.

2. Basic approximations

In terms of spherical coordinates (r, θ, ψ), the equation of the bubble surface S is considered in the form

$$r = a\{1 + \epsilon_n(t) S_n(\theta, \psi) + \delta(t)\}. \tag{4}$$

Here a is the radius of the bubble when at rest, $|\epsilon_n| \ll 1$, and S_n is a spherical harmonic of order n. The term δ independent of position on the surface will turn out to be of second order in $|\epsilon_n|$. Further terms, depending on θ and ψ, would be needed in (4) to represent the motion of the surface accurately to second order; but such terms will be seen not to affect the integal-property estimates that follow. Owing to the orthogonality and completeness provided by sets of surface harmonics, the expression (4) suffices to represent an arbitrary perturbation of the sphere $r = a$. Each term $\epsilon_n S_n$ in a summation contributes independently of the others to second-order integral properties such as δ and kinetic energy.

The spherical mean of S_n is zero for all $n \geqslant 1$, and its mean square is

$$\overline{S_n^2} = \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_0^{\pi} [S_n(\theta, \psi)]^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\psi. \tag{5}$$

The standard forms of S_n may be recalled. Thus, for axisymmetric perturbations, if $S_n = P_n(\cos\theta)$ with P_n the Legendre polynomial of order n, we obtain from (5)

$$\overline{S_n^2} = \frac{1}{2n+1}.$$

In the case of tesseral harmonics $S_n = P_n^m(\cos\theta)\cos m\psi$ $(1 \leqslant m < n)$ or sectoral harmonics with $m = n$, the result is

$$\overline{S_n^2} = \frac{1}{2(2n+1)}\frac{(n+m)!}{(n-1)!}$$

(cf. Lamb 1932, p. 118; Jeffreys & Jeffreys 1956, pp. 633, 638).

The velocity potential satisfies $\Delta\phi = 0$ everywhere in the liquid. Hence, to first order in small quantities, the kinematic boundary condition at the bubble surface (Lamb 1932, pp. 7, 474) shows that

$$\phi = -\frac{a^2\dot{\epsilon}_n}{n+1}\left(\frac{a}{r}\right)^{n+1} S_n(\theta,\psi) - \frac{a^3\dot{\delta}}{r}. \tag{6}$$

From (6), presuming $\dot{\delta}$ $(= \mathrm{d}\delta/\mathrm{d}t)$ to be $O(|\epsilon_n\dot{\epsilon}_n|)$, we easily find that the kinetic energy is given to $O(\dot{\epsilon}_n^2)$ by

$$K = \frac{2\pi}{n+1}a^5\dot{\epsilon}_n^2\,\overline{S_n^2}. \tag{7}$$

The potential energy of the bubble includes the superficial energy $T|S|$, the part of which is due to the variations in shape is $T|S|'$, where $|S|'$ is the difference $|S|-4\pi a^2$ subject to the constraint $\mathscr{V} = \frac{4}{3}\pi a^3$. From (4) it is easily found that

$$T|S|' = 2\pi(n-1)(n+2)a^2T\epsilon_n^2\,\overline{S_n^2} \tag{8}$$

(cf. Rayleigh 1879, Appendix 2). Note that, according to (4), $|S|-4\pi a^2$ includes a contribution $8\pi a^2\delta$, which too is $O(\epsilon_n^2)$ and corresponds to a volume change $\mathscr{V}' = 4\pi a^3\delta$. These effects will enter the complete second-order estimate to be evaluated in §3. Note also that $|S|' = 0$ for $n = 1$, which case describes a translation of the bubble leaving its surface area unchanged. Only the cases $n = 2, 3, \ldots$ representing changes in shape are relevant here (see LH, figure 4).

As a check on (7) and (8), these results may be used to express the frequency σ_n of infinitesimal shape oscillations, which is of course determinable by linearized theory without reference to second-order effects (cf. Lamb 1932, §275). For the simple-harmonic motion with

$$\epsilon_n = \hat{\epsilon}_n\cos\sigma_n t, \quad \dot{\epsilon}_n = -\sigma_n\hat{\epsilon}_n\sin\sigma_n t, \tag{9}$$

the mean values of K and $T|S|'$ are equal (the principle of energy partition common to normal modes of every linearized Hamiltonian system). Hence (7) and (8) give

$$\sigma_n^2 = (n-1)(n+1)(n+2)\,T/a^3, \tag{10}$$

which recovers the expression found by Lamb (1932, p. 475, equation (12)). (Recall that the density ρ of the liquid is taken to be 1; otherwise T is replaced by T/ρ in (10).)

3. Virial equation

The aim is to evaluate equation (2) to $O(\hat{\epsilon}_n^2)$. The equation shows at once that, in a state of rest with $K = 0$, $|S| = 4\pi a^2$, $\mathscr{V} = \frac{4}{3}\pi a^3$, the value of the gas pressure P_i is

$$P_{i0} = p_\infty + 2(T/a). \tag{11}$$

Because $\bar{S}_n = 0$, the definition (1) shows that W and hence $\mathrm{d}W/\mathrm{d}t$ are $O(\hat{\epsilon}_n^2, |\delta|)$.

Therefore equation (2) confirms that δ, which affects the variables $|S|$, P_i and $\mathscr{V}$ on the right-hand side, is $O(\hat{\epsilon}_n^2)$.

Let us suppose that the enclosed gas is compressed and expanded adiabatically, so that

$$P_i = P_{i0}(\mathscr{V}_0/\mathscr{V})^\gamma, \tag{12}$$

where γ is the ratio of specific heats. It follows that, to $O(|\delta|) = O(\hat{\epsilon}_n^2)$,

$$\begin{aligned}(P_i - p_\infty)\mathscr{V} - (P_{i0} - p_\infty)\mathscr{V}_0 &= -4\pi a^3\{(\gamma-1)P_{i0} + p_\infty\}\delta \\ &= -4\pi a^3\{\gamma P_{i0} - 2(T/a)\}\delta\end{aligned}$$

by (11). To this order of approximation, we also have

$$|S| = 4\pi a^2 + |S|' + 8\pi a^2\delta.$$

When these approximations are introduced on the right-hand side of (2), together with (7) for K and (8) for $T|S|'$ after (9) has been substituted for $\dot{\epsilon}_n$ and ϵ_n, the result from (2) is

$$\frac{dW}{dt} = 2\pi(n-1)(n-2)a^2 T\hat{\epsilon}_n^2\overline{S_n^2}(5\sin^2\sigma_n t - 2\cos^2\sigma_n t) - 4\pi a^3\{3\gamma P_{i0} - 2(T/a)\}\delta. \tag{13}$$

The final term in (13) can be written $-4\pi a^5\omega^2\delta$, where

$$\omega^2 = \frac{3\gamma P_{i0}}{a^2} - \frac{2T}{a^3}. \tag{14}$$

As will be confirmed below, ω is in fact the frequency of free infinitesimal oscillations in the volume of the bubble (LH, equation (2.2); Plesset & Prosperetti 1977, equation (2.8)), that is, oscillations in the mode $n = 0$ featuring a monopole potential.

To approximate dW/dt accurately to $O(\hat{\epsilon}_n^2, |\delta|)$, it is important to appreciate first why the expressions (4) for the bubble surface and (6) for ϕ are sufficient. Because their spherical mean value is zero, like that of S_n, additional terms that are $O(\hat{\epsilon}_n^2)$ make no contribution to the integral (1) defining W. For use in the approximation of W, we have

$$(\boldsymbol{X}\cdot\boldsymbol{n})\,ds = a^3\{1 + 3\hat{\epsilon}_n S_n\cos\sigma_n t + O(\hat{\epsilon}_n^2)\}\sin\theta\, d\theta\, d\psi;$$

and from (6), with (4) substituted for r, we have to $O(\hat{\epsilon}_n^2)$

$$\phi = \sigma_n a^2\hat{\epsilon}_n S_n \sin\sigma_n t\left(\frac{1}{n+1} - \hat{\epsilon}_n S_n\cos\sigma_n t\right) - a^2\dot{\delta}$$
$$+ (\text{terms } O(\hat{\epsilon}_n^2) \text{ with zero mean value}).$$

Hence (1) gives directly

$$W = 2\pi a^5\left\{\left(\frac{n-2}{n+1}\right)\sigma_n\hat{\epsilon}_n^2\overline{S_n^2}\sin 2\sigma_n t + 2\dot{\delta}\right\},$$

and so

$$\begin{aligned}\frac{dW}{dt} &= 4\pi a^5\left\{\left(\frac{n-2}{n+1}\right)\sigma_n^2\hat{\epsilon}_n^2\overline{S_n^2}\cos 2\sigma_n t + \ddot{\delta}\right\} \\ &= 4\pi a^2\{(n-1)(n-2)(n+2)T\hat{\epsilon}_n^2\overline{S_n^2}\cos 2\sigma_n t + a^3\ddot{\delta}\}.\end{aligned} \tag{15}$$

Upon the substitution of (15) for the left-hand side of (13) and upon use of the notation (14), the resulting equation reduces to

$$\ddot{\delta}+\omega^2\delta = C\sigma_n^2\{3-(4n-1)\cos 2\sigma_n t\}, \tag{16}$$

in which

$$C = \frac{1}{4(n+1)}\hat{\epsilon}_n^2\,\overline{S_n^2}. \tag{17}$$

Having the complementary function $A\sin\omega t+B\cos\omega t$, with A and B arbitrary, (16) confirms ω to be the frequency of volume pulsations. The particular integral of (16) is

$$\left.\begin{aligned}\delta &= D+E\cos 2\sigma_n t,\\ \text{with}\quad D &= 3C\frac{\sigma_n^2}{\omega^2},\quad E = (4n-1)\,C\frac{\sigma_n^2}{4\sigma_n^2-\omega^2}.\end{aligned}\right\} \tag{18}$$

Therefore, according to (6), the monopole component ϕ_0 of ϕ associated with shape oscillations at frequency σ_n is given by

$$\phi_0 = \frac{\sigma_n a^2 E\sin 2\sigma_n t}{r}, \tag{19}$$

which plainly agrees with (3) and (4). Correspondingly, the pressure fluctuation in the liquid at large distances r is

$$-\frac{\partial\phi_0}{\partial t}+O\left(\frac{1}{r^4}\right) = -\frac{2\sigma_n^2 a^3 E\cos 2\sigma_n t}{r}+O\left(\frac{1}{r^4}\right)$$

(cf. LH, equation (7.1)).

The second-order equations (18) and (19) are equivalent to the results presented in LH as (6.26), with coefficients specified by (6.24), (6.25) and (6.27). The comparative straightforwardness of the present derivation exemplifies the usefulness of the virial equation (2), which constitutes to all orders of approximation an exact relation among the integral properties of symmetric and asymmetric motions. For physical interpretations of these results, however, particularly as regards their bearing on underwater-sound generation, reference needs to be made to the full discussions in Longuet-Higgins's two papers (1989*a*, *b*).

4. Mean pressure in gas

The result (18) shows that the mean value $\bar{\delta} = D$ is positive unless $\hat{\epsilon}_n = 0$ or $\sigma_n^2/\omega^2 = 0$, the latter of which conditions holds in the limit $ap_\infty/T\to\infty$. Thus, when ω is finite, the shape oscillations cause the mean volume of the bubble to increase and consequently the mean pressure of the contained gas to decrease. The property has already been demonstrated in the previous paper B (p. 358), where the artificial case $p_\infty = 0$ was taken as an illustrative example. The conclusions can be generalized as follows.

Note first that the mean pressure $\bar{P}_i$ is reduced by shape oscillations even if the bubble is incompressible ($\mathscr{V}$ = const.). This case is represented by the limit $\omega^2/\sigma_n^2\to\infty$, which gives $D = E = 0$ in (18). But, because to $O(\hat{\epsilon}_n^2)$ the mean value $\bar{K}$ of kinetic energy still equals the mean value of $T|S|'$, the virial equation (2) shows at once that

$$\bar{P}_i-P_{i0} = -\bar{K}/\mathscr{V} < 0 \tag{20}$$

(cf. B, equation (2.21), in which $\bar{K}$ is misprinted as K).

From (12) and (18), we have to $O(\hat{\epsilon}_n^2)$

$$\bar{P}_i - P_{i0} = -3\gamma P_{i0} D = -\frac{9\gamma}{4(n+1)}\frac{\sigma_n^2}{\omega^2} P_{i0}\,\hat{\epsilon}_n^2\,\overline{S_n^2}$$

$$= -\frac{9\gamma P_{i0}\bar{K}}{4\pi\omega^2 a^5} = -\frac{3\gamma\{p_\infty + 2(T/a)\}}{\{3\gamma p_\infty + 2(3\gamma-1)(T/a)\}}\frac{\bar{K}}{\mathscr{V}_0}. \quad (21)$$

The result (20) is recovered from (21) in the limits $ap_\infty/T \rightarrow \infty$ or $\gamma \rightarrow \infty$, either of which make the bubble incompressible relative to the shape oscillations. The result given in B (equation (2.22)) is recovered when $p_\infty = 0$.

As a final practical point, it should be acknowledged that the adiabatic law (12) becomes unreliable for air bubbles in water that are small enough for their resonance frequency $\omega/2\pi$ to be a few kHz or more. The pressure–volume relation becomes progressively closer to the isothermal law ($\gamma = 1$ in (12)) as bubble size decreases (for a review of this aspect, see Plesset & Prosperetti 1977, pp. 148, 149). The crucial dimensionless parameter in this regard is $(\omega/\kappa)^{\frac{1}{2}}a$, where κ is the thermal diffusivity of the gas. This parameter, which is the quotient of a and the effective diffusion length for one period $2\pi/\omega$, decreases with a (cf. the expression (14) for ω). When it is $O(1)$ or less, heat exchange between the gas and liquid is rapid enough for approximately isothermal conditions to prevail. The present results (18), (19) and (21) remain applicable, of course, if γ is replaced by 1 or any other value between 1 and the specific-heat ratio.

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